Optimal Blending and Production

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Abstract

Linear programming is used to treat the combined blending and production problem. The production problem is set up in such a way that it includes nearly all the operations such as distillation, extraction, and even reaction in a processing industry. Since these operations are usually nonlinear it is necessary to linearize them for linear programming. This is done by choosing a set of typical operating points. The solution can be made arbitrarily close to the optimum by repeatedly solving the problem with different sets of typical operating points. A simple example is used to illustrate the approach and the technique is compared with other linearization techniques.

Introduction

Linear programming has proven to be one of the most versatile and practical techniques for solving large complex linear optimization problems [1]. One of the traditional applications of this technique is the blending problem. In this problem the objective is to determine the least cost or most profitable blend from the infinite number of possible blends which satisfies a given number of restrictions and specifications. The problem of gasoline blending is a well publicized example [1-3]. The dairy industry has the same type of problem in their feed blending problems. Another example is the textile industry. For instance the raw cotton or synthetic material used in the initial yarn blend may come from a number of sources. These have to be blended together considering their respective qualities to come up with a uniform blend that can be efficiently processed through the production facilities and yield an uniform yarn. The manufacturers of finished plastic products such as plastic cups have the same blending problem.

Another frequent application of linear programming is to the problem of production control. In this problem there is a variety of different products each competing for limited amounts of raw material and plant capacities. The objective is to find the optimal combination of products that should be produced which will maximize the overall profit and best utilize the available resources. In some problems where the process cannot be expressed by linear relationships, linear approximations are made and "good" answers can still be obtained by using linear programming. An example in production control is the optimization of refinery operations by linear programming [3-5].

In some situations we are faced with both production control and blending within the same problem. For example the situation may occur where the least cost blend may be very expensive to process whereas a more expensive blend may be processed cheapter resulting in more overall profit. It is this problem to which this discussion is addressed. Thus the purpose of this paper is to develop a generalized linear programming model of the combined blending and production problem. Since some of these processes have nonlinear relationships, it is also illustrated how these relationship can be incorporated into a linear programming model. This generalization is applicable to a variety of processes.

Linear Programming

To illustrate the essence of the linear programming technique, let us consider a simple two-dimensional production problem. A plant can produce two products, P_1 and P_2. Because of market conditions only a maximum of $P_1 = 350$ and $P_2 = 300$ can be sold. From a prior commitment, at least 150 units of P_1 must be produced. The capacity of the plant is 500 units of P_1 and P_2. That is the combination of P_1 and P_2 must not exceed 500. The profit from P_1 is \$1.00 and the profit from P_2 is \$2.00. The problem is to determine the combination of P_1 and P_2 which maximizes the profit and which does not violate any constraints.

The constraints can be represented by the following equations:

$$
\begin{aligned}
P_1 \qquad & \leq 350 \\
P_1 \qquad & \geq 150 \\
P_2 \qquad & \leq 300 \\
P_1 + P_2 \quad & \leq 500
\end{aligned}
$$

where the first two equations represent the upper and lower production limits on P_1 and the third equation represents the upper limit on P_2. The last equation represents the plant capacity. The profit function to be maximized is

$$z = P_1 + 2P_2$$

where z is the profit in dollars.

The solution of this linear programming problem is illustrated in Figure 1. The shaded area represents the feasible region where all four of the inequality constraints are satisfied. The optimal solution is

$$P_1 = 200, \quad P_2 = 300, \quad z = \$800$$

As can be seen from Figure 1, one important property of the linear programming problem is that the optimal solution must lie in at least one of the extreme points which are ~~interceptions~~ intersections of the constraints. Thus, the optimum of an linear programming problem can be obtained by searching the extreme points only. The simplex method [1] is essentially a systematic way for searching the extreme points for large linear programming problems. By the use of this systematic search technique, extremely large problems with several thousands variables can be solved with current computers and computer program packages that are available for solving linear programming problems.

The Blending and Production Problem

Managers are constantly faced with the problem of blending raw materials and determining how to control the plant so that the overall profit is maximized and the product meets the required specifications. More specifically some of the decisions which must be made are:

1. What quantities of the limited raw materials should be put into the blend?
2. At which mode of operations should the plant be operated considering the capacities, recycle and the material loss or gain from the process?
3. How should questions 1. and 2. be answered so that the overall profit is maximized and the product meets its specifications.

In developing a linear programming model it is necessary to determine which are the important variables that must be considered. Usually these variables depend on which parameters can be controlled and their importance relative to profit.

For the purpose of illustration, a simple problem is solved in the next section. The process considered in this example incorporates recycle, and the changes in the qualities and quantities of the raw material due to processing. This simple model can be extended easily to include more complex production processes such as chemical reaction and separation processes.

An Example

To illustrate this approach, a simple but certainly not trivial example will be discussed. It will be seen that this example possesses most of the characteristics of a processing industry and can be extended easily to cover other situations which occur in a variety of processes.

Three raw materials R_1, R_2, and R_3, are to be blended and processed into a finished product. In this problem we assume that the three raw materials are three different grades of plastic powder. These different grades of plastic raw material are to be blended and molded into a finished product which must possess two properties or qualities. Quality 1 is a desirable property and quality 2 is undesirable. For example, quality 1 may be the ability to withstand heat and quality 2 may be an undesirable odor. The finished product must possess at least 220 units of quality 1 and not more than 40 units of quality 2. There is a market for all the product produced at $0.40/lb. The qualities of the three grades raw material together with their costs are listed in Table 1.

The three raw materials are blended first (see Figure 2). The resulting blend is then processed in a processing unit where the operating range is known. Depending upon the particular operating conditions used within this operating range, processing of the blend may increase or decrease the qualities and quantities of the original raw material. The operating cost per pound of material processed also depends on the operating condition used. Since there is only one piece of equipment available for processing and also since a change in the operating condition is time consuming and impractical, we shall assume that all the raw materials must be processed under the same operating condition. The problem is to select the proper amounts of the different grades of raw materials to blend and to select the operating condition so that the profit is maximized.

In general, the increase or decrease in the qualities and quantities of the raw material and the processing cost depend upon the operating conditions nonlinearly. It is not a simple matter to select the proper operating condition. To avoid this difficulty, three typical operating conditions or settings S_1, S_2, and S_3 are selected. For example, S_1 may be the lower limit of the operating range, and S_3 the upper limit. The processing costs and the changes in qualities and quantities of the original raw material under the three typical settings and listed in Table 2. If the solution results in a split between two settings, that is part of the material is processed at one operating condition and the other part is processed at a different one, the problem is rerun with additional settings until all the material is processed under one operating condition.

Because of different recycle rates or different processing times for the different settings the capacity of the unit is different for each of the settings. This change in capacity is listed in the last row of Table 2 under the heading "recycle". If setting 2 is used, the unit has a capacity of 100 lbs./hour. If the unit were operated on a batch basis, the capacity would be in pounds per batch. The unit has a capacity of 100/1.1 lbs./hour for setting 1 and 100/0.92 lbs./hour for setting 3.

The problem is to determine the proper combination of raw materials and setting so that the following profit equation is maximized.

$$\begin{aligned} z = 0.40p - 0.31x_1 - 0.28x_2 - 0.35x_3 \\ - 0.132y_1 - 0.121y_2 - 0.110y_3 \end{aligned} \tag{1}$$

where p is the total product, lbs./hr.; x_1, x_2, and x_3 are the amount of raw materials R_1, R_2, and R_3 used, respectively, lbs./hr.; y_1, y_2, and y_3 are the material processed in S_1, S_2, and S_3 respectively, lbs./hr; and z is the profit in dollars/hour.

Since the finished product must possess the required qualities, the following constraints must be satisfied

$$220x_1 + 200x_2 + 245x_3 = q_1 \tag{2}$$

$$48x_1 + 58x_2 + 38x_3 = q_2 \tag{3}$$

$$0.9q_1 + 0.0y_1 + 0.0y_2 + 0.0y_3 \geq 220p \tag{4}$$

$$0.9q_2 - 18y_1 - 13y_2 - 7y_3 \leq 40p \tag{5}$$

where q_1 and q_2 are qualities 1 and 2 of the total raw material, respectively, lbs-units/hr. The coefficients of 0.9 for the q's in Equations (4) and (5) are used to take care of the loss in the amount of raw material due to processing. Equations (2) and (3) represent the total qualities of the raw materials used. Equations (4) and (5) represent the requirement of the two qualities on the finished product.

Material balances of the raw material in the blending unit, the material between the blend and the processing units, and the material in the processing unit give

$$x_1 + x_2 + x_3 = X \tag{6}$$

$$y_1 + y_2 + y_3 = 0.9X \tag{7}$$

$$y_1 + y_2 + y_3 = p \tag{8}$$

where X represents the total amount of raw materials used, lbs/hr.

Finally, the amount of material processed must not exceed the capacity of the processing unit

$$1.1y_1 + 1.0y_2 + 0.92y_3 \leq 100 \tag{9}$$

The problem is to find the values of p, x's, and y's so that Equation (1) is maximized subject to the constraints of Equation (2) through (9). Since all these equations are linear, this problem can be solved easily by the simplex algorithm. The initial tableau of the linear programming model for this problem is illustrated in Table 3. The maximum profit for this example is $z = 31.04$ dollars/hr. The other results are

$$x_1 = 108.69565 \text{ lbs./hr}$$

$$x_2 = 0.0$$

$$x_3 = 12.07729 \text{ lbs./hr}$$

$$y_1 = y_2 = 0$$

$$p = y_3 = 108.69565 \text{ lbs./hr.}$$

Generalizations

The problem solved in the previous section can be generalized in various ways. Instead of one finished product, two or more products with different qualities may be produced from the same raw materials. Each product can again be produced by the use of different operating conditions.

There are roughly three different classes of operations in the processing industry. The simplest one is the one considered in the previous section, namely, the processing of raw plastic material into finished product. Another general class of operation is the separation operations such as distillation and extraction. The most complex class of operations is chemical reaction. The linear programming model treated in the previous section can be used to consider any of these operations. The only difference is that instead of one product, several products may be produced. Furthermore, some of the raw materials in the feed may disappear in the product due to chemical reaction. Thus, the general linear programming model which would include separation and reaction would be more complex but, could still be solved by linear programming. It should be emphasized that this general model would not and cannot include the detailed reaction or separation process due to the nonlinearity relationships. Only the more general aspects of the problem such as material balances and processing yields can be considered. The processing costs, which are generally nonlinear, must again be treated by the selection of representative operating conditions or settings as we have done in the previous example.

Discussion

The linear programming model discussed in the previous sections should be a very useful tool for production management and planning of complex processing plants. The use of a few discrete operating points to represent a continuous nonlinear operating range should be a very useful idea. It is true that the optimum obtained is most probably not the true optimum due to this discrete representation. However, the solution can be made arbitrarily near to the true optimum by repeatedly solving the same problem with different operating points.

There are other techniques to treat nonlinearities by linear programming. One of these techniques is to use the evolutionary operations concept which is known as EVOP [6,7]. The technique is essentially a search technique in which we sample over various operating ranges in the process and hopefully converge to an optimal operating condition. The operating ranges selected are fairly small so that linear relationships can be used.

The second technique that is generally used for nonlinear problems is a technique in which the nonlinear equations are repeatedly linearized by Taylor series and the linearized equations are then solved by linear programming. An elaborate computer program known as POP has been set up for this technique. The idea of this technique is to have a nonlinear solution space and imbed a linear programming problem within it. We then search the possible regions with our linear programming model and this is done by linearizing the nonlinear restraints over a small region. As the model is moved from one region to the other these restraints changed so that the nonlinear restrictions are represented fairly accurately. POP has been used in the processing industry. Generally, POP is more restrictive and not very easy to apply in large practical problems. The difficulty principally lies in the iterative process which may not converge at all.

Actual experience has shown that the use of a set of typical operating points to represent a nonlinear range is much more practical than the POP program.

The main advantage of linear programming is its ability to handle extremely large, complex, linear programs. A problem as large as thousand restrictions and several thousand variables can be solved using linear programming with the current computers and computer programs that are available.

References

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Table 1. Raw Material

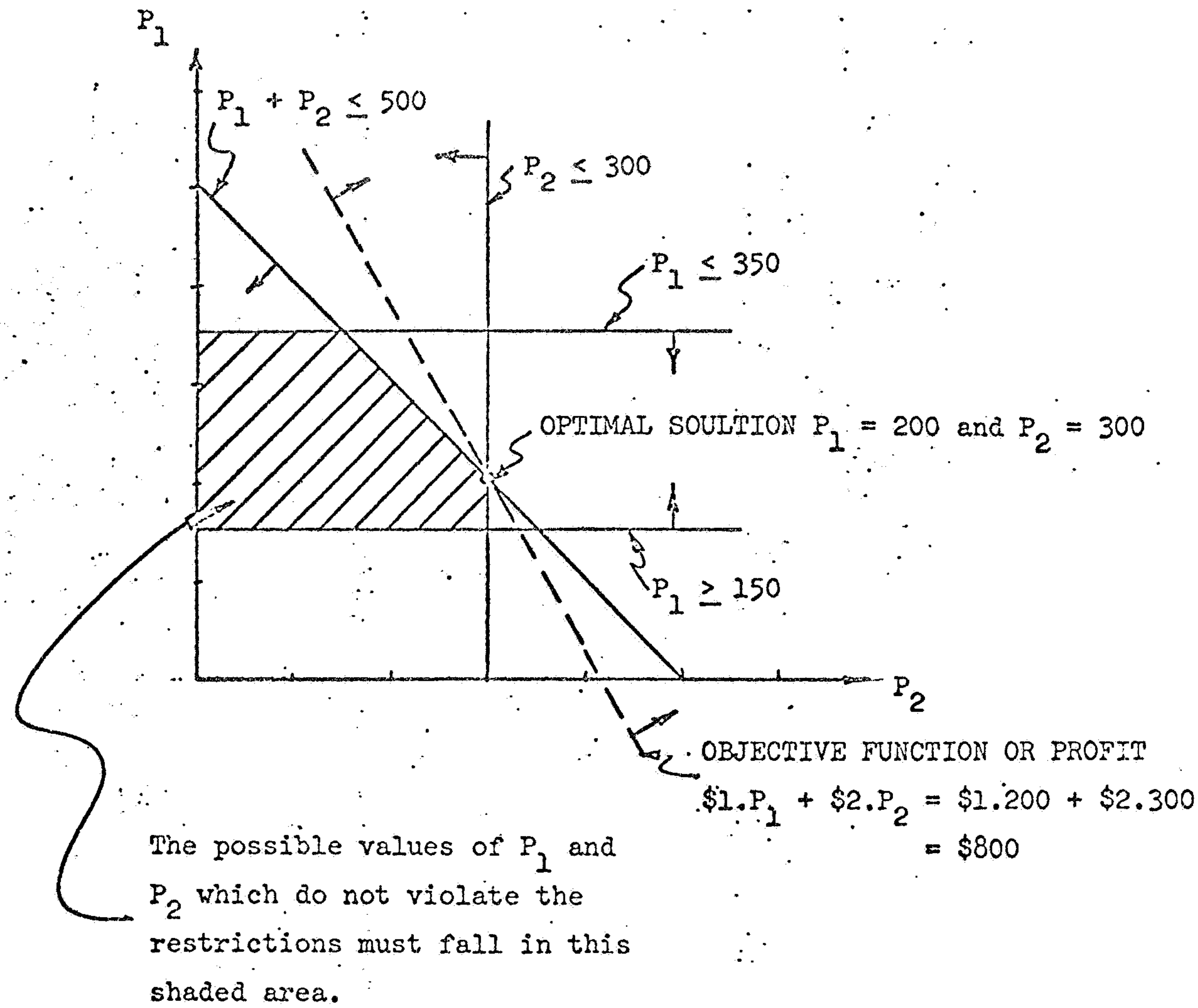
	R_1	R_2	R_3
Quality 1, unit/lb	220	200	245
Quality 2, unit/lb	48	58	38
Cost $/lb	.31	.28	.35

Table 2. Operating Settings

	S_1	S_2	S_3
Changes in Quality 1, unit/lb	0	0	0
Changes in Quality 2, unit/lb	-18	-13	-7
loss due to processing, lb/lb feed	0.1	0.1	0.1
Cost, $/lb	0.132	0.121	0.110
Recycle (S_2 base)	1.1	1.0	0.92

		R1	R2	R3	Q1	Q2	Total Raw Matl	S1	S2	S3	PRODUCT		
1	Profit	-.31	-.28	-.35				-.132	-.121	-.110	.40	=	Max
2	Quality 1	220	200	245	-1							=	0
3	Quality 2	48	58	38		-1						=	0
4	Matl Bal	+1	+1	+1			-1					=	0
5	Quality 1 TR				.9			0	0	0	-220	$\geq$	0
6	Quality 2 TR					.9		-18	-13	-7	-40	$\leq$	0
7	Raw Matl TR						-.9	1	1	1		=	0
8	Capacity							1.1	1.0	.92		$\leq$	100
9	Total Product							1	1	1	-1	=	0

Table 3. Linear Programming Model



P_1
$P_1 + P_2 \leq 500$
$P_2 \leq 300$
$P_1 \leq 350$
OPTIMAL SOULTION $P_1 = 200$ and $P_2 = 300$
$P_1 \geq 150$
P_2
OBJECTIVE FUNCTION OR PROFIT
$\$1.P_1 + \$2.P_2 = \$1.200 + \$2.300$
$= \$800$
The possible values of P_1 and P_2 which do not violate the restrictions must fall in this shaded area.

Figure 1

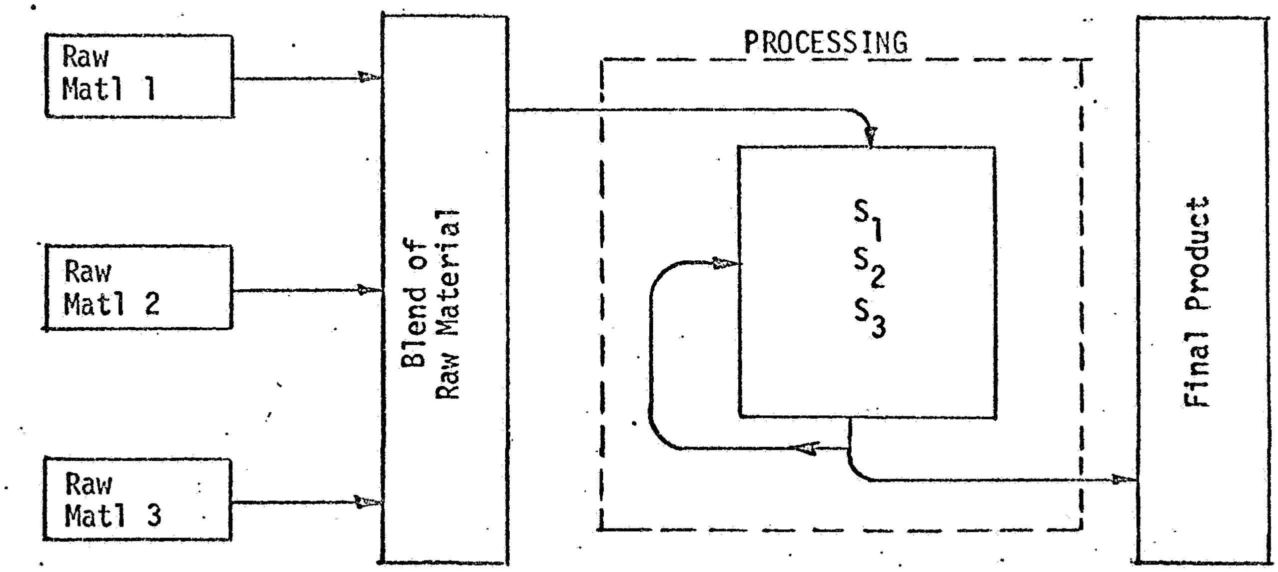


Figure 2. Flow Diagram